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# Dirac's Conjecture

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An integral invariant for a singular Lagrangian which depends on time explicitly is deduced. The connection between this invariant and Dirac's conjecture is discussed. An example shows that Dirac's conjecture fails.

Dirac's generalized canonical formalism for a singular Lagrangian plays an important role in modern field theories. In spite of the achievements of this theory, there are some basic problems (Lusanna, 1991). One of them is Dirac's conjecture (Dirac, 1964). Dirac in his work on the generalized canonical formalism conjectured that all first-class constraints generate gauge transformations. If this conjecture holds true, then the dynamics of a constrained Hamiltonian system possessing primary $\{\phi_n\}$ and secondary $\{\chi_m\}$ first-class constraints should be correctly described by the equations of motion arising from the extended Hamiltonian $H_E$ (Costa *et al.*, 1985). From time to time there have been objections to Dirac's conjecture (Cawley 1979, 1980; Frenkel, 1980; Castellani, 1982; Qi, 1990; Li and Li, 1991; Li, 1991). In many physically important systems Dirac's conjecture does not lead to a wrong result. In spite of the lack of a proof of this conjecture (Galvão and Boechat, 1990) we shall further discuss this problem from the viewpoint of the generalized Poincaré-Cartan integral invariant (GPCII) for a singular Lagrangian.

The GPCII plays an important role in mechanics and field theories. This invariant had been generalized to singular Lagrangians (Benavent and Gomis, 1979; Dominici and Gomis, 1980, 1982). These authors discussed only the case where the constraints do not depend on time explicitly. Here a slight modification is made to consider that a system whose Lagrangian

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$L(t, q, \dot{q})$ ($q = [q^1, q^2, \ldots, q^n]$) is singular and depends on time $t$ explicitly. Due to the singularity of the Lagrangian, this system is subject to some inherent phase constraints. Let

$$\Lambda_\alpha(t, q, p) \approx 0 \qquad (\alpha = 1, 2, \ldots, A) \tag{1}$$

be primary first-class constraints ($p = [p_1, p_2, \ldots, p_n]$) and

$$\theta_m(t, q, p) \approx 0 \qquad (m = 1, 2, \ldots, M) \tag{2}$$

be second class, where the sign $\approx$ (weak equality) means equality on the hypersurface $\Gamma$ defined by equations (1) and (2). The canonical equations for this system can be written as (Sundermeyer, 1982)

$$\dot{q}^i \approx \frac{\partial H_c}{\partial p_i} + \lambda^\alpha \frac{\partial \Lambda_\alpha}{\partial p_i} - \frac{\partial \theta_m}{\partial p_i} C^{-1}_{mm'} \left[ \{\theta_{m'}, H_c\} + \frac{\partial \theta_{m'}}{\partial t} \right] \tag{3a}$$

$$\dot{p}_i \approx -\frac{\partial H_c}{\partial q^i} - \lambda^\alpha \frac{\partial \Lambda_\alpha}{\partial q^i} + \frac{\partial \theta_m}{\partial q^i} C^{-1}_{mm'} \left[ \{\theta_{m'}, H_c\} + \frac{\partial \theta_{m'}}{\partial t} \right] \tag{3b}$$

where $C_{mm'} = \{\theta_m, \theta_{m'}\}$, $\{\cdot, \cdot\}$ denotes the Poisson bracket, $H_c$ is a canonical Hamiltonian, $\lambda^\alpha(t)$ are Lagrange multipliers.

Let us consider the infinitesimal transformation in extended phase space

$$\begin{aligned} t &\to t' = t + \Delta t(\theta) \\ q^i(t) &\to q'^i(t') = q^i(t) + \Delta q^i(t, \theta) \\ p_i(t) &\to p'_i(t') = p_i(t) + \Delta p_i(t, \theta) \end{aligned} \tag{4}$$

where $\theta$ is a parameter which satisfies

$$q^i(t, 0) = q^i(t), \qquad p_i(t, 0) = p_i(t) \tag{5}$$

Under the transformation (4) the variation of the canonical action is given by (Li and Li, 1991)

$$\begin{aligned} \Delta I_p = I'_p(\theta)\, \Delta(\theta) &= \Delta \int_{t_1}^{t_2} (p_i \dot{q}^i - H_c)\, dt \\ &= \int_{t_1}^{t_2} \left\{ \left( \dot{q}^i - \frac{\partial H_c}{\partial p_i} \right) \delta p_i - \left( \dot{p}_i + \frac{\partial H_c}{\partial q^i} \right) \delta q^i \right. \\ &\qquad \left. + D[p_i \delta q^i + (p_i \dot{q}^i - H_c)\, \Delta t] \right\} dt \end{aligned} \tag{6}$$

Let the simultaneous variations $\delta q^i = \Delta q^i - \dot{q}^i \Delta t$ and $\delta p_i = \Delta p_i - \dot{p}_i \Delta t$

satisfy the following conditions:

$$\frac{\partial \Lambda_\alpha}{\partial p_i}\delta p_i+\frac{\partial \Lambda_\alpha}{\partial q^i}\delta q^i\approx 0 \tag{7}$$

$$\frac{\partial \theta_m}{\partial p_i}\delta p_i+\frac{\partial \theta_m}{\partial q^i}\delta q^i\approx 0 \tag{8}$$

Using the Lagrange multipliers $\lambda^\alpha(t)$ and $\lambda^m(t)$ and combining the expressions (6)–(8), one obtains

$$\Delta I_p=I'_p(\theta)\,\Delta\theta=\int_{t_1}^{t_2}\left[\left(\frac{\delta I_p}{\delta p_i}+\lambda^\alpha\frac{\partial \Lambda_\alpha}{\partial p_i}+\lambda^m\frac{\partial \theta_m}{\partial p_i}\right)\delta p_i+\left(\frac{\delta I_p}{\delta q^i}+\lambda^\alpha\frac{\partial \Lambda_\alpha}{\partial q^i}+\lambda^m\frac{\partial \theta_m}{\partial q^i}\right)\delta q^i+D(p_i\,\Delta q^i-H_c\,\Delta t)\right]dt \tag{9}$$

From the stationary of the constraints, one gets (Sundermeyer, 1982)

$$\lambda_m\approx -C^{-1}_{mm'}\left[\{\theta_{m'},H_c\}+\frac{\partial\theta_{m'}}{\partial t}\right] \tag{10}$$

Substituting (10) into (9) and using the canonical equations of a constrained system, one obtains

$$\Delta I_p=I'_p(\theta)\,\Delta\theta=[p_i\,\Delta q^i-H_c\,\Delta t]_1^2 \tag{11}$$

In the extended phase space spanned by the variables $q^i$, $p_i$, and $t$, one can choose any simple closed curve $C$ lying in the subspace $\Gamma$ of the extended phase space defined by the constraints. Taking the integral of the expression (11) along the curve $C$, one obtains that the integral

$$J=\oint_c [p_i\,\Delta q^i-H_c\,\Delta t]=\text{inv} \tag{12}$$

is invariant with an arbitrary displacement (with deformation) of the contour $C$ along any tube of dynamical trajectories (Gantmacher, 1970). The integral (12) is called the GPCII for a singular Lagrangian.

By following analogous reasoning (Dominici and Gomis, 1980) and starting from the GPCII, it is possible to show that the canonical equations of a system with first-class constraints $\{\phi_\alpha\}$ and second-class constraints $\{\theta_m\}$ are given by

$$\dot q^i\approx\frac{\partial H_c}{\partial p_i}+\lambda^\alpha\frac{\partial\phi_\alpha}{\partial p_i}-\frac{\partial\theta_m}{\partial p_i}C^{-1}_{mm'}\left[\{\theta_{m'},H_c\}+\frac{\partial\theta_{m'}}{\partial t}\right] \tag{13a}$$

$$\dot p_i\approx-\frac{\partial H_c}{\partial q^i}-\lambda^\alpha\frac{\partial\phi_\alpha}{\partial q^i}+\frac{\partial\theta_m}{\partial q^i}C^{-1}_{mm'}\left[\{\theta_{m'},H_c\}+\frac{\partial\theta_{m'}}{\partial t}\right] \tag{13b}$$

The GPCII for a singular Lagrangian is a useful tool for the discussion of Dirac's conjecture. In the above derivation of the GPCII (12), the primary first-class constraints only have been taken into account. If Dirac's conjecture holds true, the dynamics of the system is generated by an extended Hamiltonian $H_E$ which is obtained by adding a linear combination of all secondary first-class constraints to the primary ones. As long as the simultaneous variations of the canonical variables satisfy the conditions (7) and (8) for all first- and second-class constraints, the GPCII (12) can also be deduced from the equations of motion arising from an extended Hamiltonian. Thus, we conclude that the necessary and sufficient condition for (13) to be canonical equations arising from $H_E$ is that the GPCII exists for such a system. Making use of the GPCII enables us to write the equations of motion for a singular Lagrangian as canonical equations arising from $H_E$; we see that all first-class constraints appear in the Hamiltonian; one cannot introduce any distinction between them. That is, the existence of the GPCII for a system is equivalent to that Dirac's conjecture holds true for such a system. The GPCII differs from usual ones for a regular Lagrangian in that the variations of canonical variables are not independent but satisfy the conditions (7) and (8). Therefore, for a system with a singular Lagrangian, whether the GPCII exists and all the canonical equations determined by $H_E$ can be derived from the GPCII may be considered as a criterion for validity of Dirac's conjecture. If in a given case the GPCII does not exist or if it exists but the canonical equations arising from the extended Hamiltonian $H_E$ cannot all be derived from it, then Dirac's conjecture fails in this case.

To illustrate this result, let us consider a model whose Lagrangian is given by

$$L = \dot{x}\dot{z} + xz - y\dot{z} \tag{14}$$

The Euler–Lagrange equations for this example are given by

$$\ddot{z} - z = 0, \qquad \dot{z} = 0, \qquad \ddot{x} - x - \dot{y} = 0 \tag{15}$$

We see that $z = 0$, $y$ is left arbitrary, and $x$ is related to $y$ by the third equation of motion. This Lagrangian describes a system constrained in the $x$–$y$ plane, where the motion is not arbitrary in both directions, but has to satisfy the third equation of motion.

The canonical momenta conjugate to $x, y, z$ are

$$p_x = \dot{z}, \qquad p_y = 0, \qquad p_z = \dot{x} - y \tag{16}$$

respectively. The canonical Hamiltonian is given by

$$H_c = p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - L = p_xp_z - xz + yp_x \tag{17}$$

with the primary constraint $\phi = p_y \approx 0$. The total Hamiltonian is given by

$H_T = H_C + \lambda\phi$, where $\lambda(t)$ is a Lagrange multiplier. The stationarity condition of primary constraint, $\{\phi, H_T\} \approx 0$, yields the secondary constraint $\chi_1 = p_x \approx 0$. The stationarity of the secondary constraint $\chi_1$ yields another secondary constraint $\chi_2 = z \approx 0$. All these constraints are first class. The extended Hamiltonian is given by $H_E = H_C + \lambda_1\phi + \mu_1\chi_1 + \mu_2\chi_2$, where $\lambda_1(t)$, $\mu_1(t)$, and $\mu_2(t)$ are Lagrange multipliers. If this system has the GPCII and the variations of the canonical variables satisfy the conditions (7) and (8), then the secondary constraint $\chi_2 \approx 0$ is invariant under the variations of canonical variables, which implies $\Delta z \approx 0$. Due to this restriction on canonical variables in the GPCII, one cannot deduce all the canonical equations (determined by $H_E$) using the GPCII. This means that the equivalence between the GPCII and the canonical equations (determined by $H_E$) is violated. That is, the GPCII does not exist under the condition $\Delta z \approx 0$ and Dirac's conjecture fails in this example.

This conclusion can also be obtained from another point of view (Li, 1993). The discussion for other counterexamples (Cawley, 1979, 1980; Frenkel, 1980) can proceed in the same way. The above example differs from those of Lusanna (1991), including all relevant so-called "pathological examples." The total Hamiltonian is differentiable on the manifold of constraints and there is no linearization of the constraints in the above example.

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